Apr 7: Symmetric functions and Lagrange's solution of the quartic

Outline

- Reviewing symmetric functions
- Fund thm of symmetric functions
- Lagrange's soln to the quartic

§1 Symmetric functions

Recall that S_n acts on poly. ring $k[x_1, \dots, x_n]$ via:

$\sigma \in S_n$, $f \in k[x_1, \dots, x_n]$

$$(\sigma \cdot f)(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

Ex: $n=4$ $f = x_1^3 x_2 + x_3^2 x_4^2$

$$\sigma = (1\ 2\ 3)$$

$$\sigma \cdot f = x_2^3 x_3 + x_1^2 x_4^2$$

We say $f(x_1, \dots, x_n)$ symmetric if $\sigma \cdot f = f$ $\forall \sigma \in S_n$.

Equivalently, stabilizer $G_f = S_n$

$\iff$ orbit $Gf = \{f\}$

DEFN The elementary symmetric functions are:

$$s_1 = x_1 + \dots + x_n$$

$$s_2 = x_1 x_2 + x_1 x_3 + \dots + x_{n-1} x_n$$

$$\vdots$$

$$s_k = \sum_{1 \le i_1 < \dots < i_k \le n} x_{i_1} \cdots x_{i_k}$$

$$s_n = x_1 x_2 \cdots x_n$$

These are all symmetric!

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$$s_k = \sum_{1 \le i_1 < \cdots < i_k \le n} x_{i_1} \cdots x_{i_k}$$

$$s_n = x_1 x_2 \cdots x_n$$

FACT: If $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ factors as $(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$

$$a_{n-1} = -(\alpha_1 + \alpha_2 + \cdots + \alpha_n) = -s_1(\alpha_1, \ldots, \alpha_n)$$

$$a_{n-2} = s_2(\alpha_1, \ldots, \alpha_n)$$

$$a_{n-3} = -s_3(\alpha_1, \ldots, \alpha_n)$$

$$\vdots$$

$$a_0 = (-1)^n s_n(\alpha_1, \ldots, \alpha_n)$$

Rmk: Some people index

$$f = x^n + a_1 x^{n-1} + \cdots + a_n$$

Upshot: Coefficients of f are symmetric functions in roots α_i.

Strategy: Try to ~~solve for~~ the roots α_i using properties of symmetric functions.

FUND THM OF SYMMETRIC FUNCTIONS

Any symmetric function $f(x_1, \ldots, x_n)$ can be written as

$$f(x_1, \ldots, x_n) = g(s_1, \ldots, s_n)$$

for some poly. g.

Fund Thm of Symmetric Functions

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for some poly. g.

Example: $f = x_1^3 + x_2^3 + x_3^3$

How can write f in terms of

$s_1 = x_1 + x_2 + x_3$ deg 1

$s_2 = x_1x_2 + x_1x_3 + x_2x_3$ $\boxed{2}$

$s_3 = x_1x_2x_3$? $\boxed{3}$

Look at

$s_1^3 = (x_1+x_2+x_3)^3$

$s_1s_2 = (x_1+x_2+x_3)(x_1x_2+x_1x_3+x_2x_3)$

$s_3 = x_1x_2x_3$

Ex: Take $f = s_1$

$g(x_1, .. x_n) = x_1$ ✓

Try to write

$x_1^3 + x_2^3 + x_3^3 = a s_1^3 + b s_1s_2 + c s_3$

for some a, b, c.

- Look at x_1^3

LHS coeff $= 1$ $\boxed{a=1}$

RHS coeff $= a$

$\rightsquigarrow \underbrace{x_1^3+x_2^3+x_3^3}_{\text{coeff } 0} - \underbrace{s_1^3}_{3} = b\underbrace{s_1s_2}_{1} + c\underbrace{s_3}_{0}$

- Look at $x_1^2x_2$ $\boxed{b=-3}$

$\rightsquigarrow \underbrace{(x_1^3+x_2^3+x_3^3)}_{0} - (\underbrace{s_1^3}_{6} - 3\underbrace{s_1s_2}_{3}) = c \cdot s_3$

$-6+9 = c$

- Look at $x_1x_2x_3$ $\boxed{c=3}$

Fund Thm of Symmetric Functions

Any symmetric function $f(x_1, .. x_n)$ can be written as

$$f(x_1, .., x_n) = g(s_1, .., s_n)$$

for some poly. g.

Proof strategy: Look at monomials in $x_1 .. x_n$

- Choose lexigraphical ordering $x_1^3 > x_1^2 x_2 > \cdots$
- See proof in notes

Lagrange's solution to the quartic

By similar reductions to the cubic, can assume that

$$f = x^4 + a_2 x^2 + a_1 x + a_0$$

(in notes a_3 a_4)

Suppose $f(x) = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)(x-\alpha_4)$

$$\boxed{0 = a_3 = -(\alpha_1 + \cdots + \alpha_4)} = -s_1$$

$a_2 = s_2$ $\qquad \alpha_3 + \alpha_4 = -(\alpha_1 + \alpha_2)$

$a_1 = -s_3$

$a_0 = s_4$

Let $f_1 = (\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4)$

not symmetric

$\rightsquigarrow$ The orbit under S_4

$$\begin{cases} f_1 \\ f_2 = (\alpha_1 + \alpha_3)(\alpha_2 + \alpha_4) \\ f_3 = (\alpha_1 + \alpha_4)(\alpha_2 + \alpha_3) \end{cases}$$

Main insight: • First solve for f_i's. ✓
• Then solve for α_i's.

$$\boxed{(x-f_1)(x-f_2)(x-f_3) =}$$
$$\underbrace{x^3 - (f_1+f_2+f_3)x^2 + (f_1f_2+f_1f_3+f_2f_3)x + f_1f_2f_3}_{g(x) =}$$

HW

symmetric
$$\begin{cases} f_1 + f_2 + f_3 = 2s_2 \\ f_1f_2 + f_1f_3 + f_2f_3 = ? \\ f_1f_2f_3 = ? \end{cases}$$

Reminder: We know s_i's!

Since $g(x)$ is a cubic, we can solve it!

$\rightsquigarrow$ Gives us the f_i's!

Know $\alpha_3 + \alpha_4 = -(\alpha_1 + \alpha_2)$

$f_1 = -(\alpha_1 + \alpha_2)^2 \qquad \alpha_1 + \alpha_2 = \sqrt{-f_1}$

Since $g(x)$ is a cubic, we can solve it!

$\rightarrow$ Gives us the f_i's!

Know $\alpha_3+\alpha_4 = -(\alpha_1+\alpha_2)$

$f_1 = -(\alpha_1+\alpha_2)^2 \quad \alpha_1+\alpha_2 = \sqrt{-f_1}$

Get eqns

$\alpha_1+\alpha_2 = \sqrt{-f_1} \qquad \alpha_1+\alpha_3 = \sqrt{-f_2} \qquad \alpha_1+\alpha_4 = \sqrt{-f_3}$

$\alpha_3+\alpha_4 = -\sqrt{-f_1} \qquad \alpha_2+\alpha_4 = -\sqrt{-f_2} \qquad \alpha_2+\alpha_3 = -\sqrt{-f_3}$

Solve these linear eqns

$$\alpha_1 = \frac{\sqrt{-f_1}+\sqrt{-f_2}+\sqrt{-f_3}}{2}$$

$$\alpha_2 = \frac{\sqrt{-f_1}-\sqrt{-f_2}-\sqrt{-f_3}}{2}$$

$\vdots$